Symmetric solutions to dispersionless 2D Toda hierarchy, conformal dynamics and Hurwitz numbers

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Conformal dynamics

Let $\mathcal{H}$ be the set of all compact simply-connected domains $\infty \in \mathrm{Q} \subset \overline{\mathbb{C}} \setminus 0$ whose boundary is a smooth curve $\gamma = \partial \mathrm{Q}$. It is follow from Riemann theorem that for any $Q \in \mathcal{H}$ there exists a single biholomorphic map

$$w_Q(z) : Q \to \Lambda = \{z \in \overline{\mathbb{C}} | |z| > 1\}$$

such that $w_Q(\infty) = \infty$ and $w_Q'(\infty)$ — real positive.

The set of moments of the domain Q

$$t_0(Q) = \frac{1}{\pi} \iint\limits_{\overline{\mathbb{C}} \setminus Q} dx\, dy \quad t_k(Q) = -\frac{1}{\pi k} \iint\limits_{Q} z^{-k}\, dx\, dy\,, \quad k \geq 1$$

are local coordinates on $\mathcal{H}$ (P.Etingof, A.Varchenko 1992).
Put $\mathbf{t}(Q) = (t_1(Q), t_2(Q), \dots,)$ and $\bar{\mathbf{t}}(Q) = (\bar{t}_1(Q), \bar{t}_2(Q)\ , \dots,)$.

According to P.Wiegmann and A.Zabrodin (2000)

$$w_Q(z) = z\exp\left(\left(-\frac{1}{2}\frac{\partial^2}{\partial t_0^2} - \frac{\partial}{\partial t_0}\sum_{k\geq 1}\frac{z^{-k}}{k}\frac{\partial}{\partial t_k}\right)F_{con}\right)(t_0, \mathbf{t}(Q), \bar{\mathbf{t}}(Q)),$$

where $F_{con}(t_0, \mathbf{t}, \bar{\mathbf{t}})$ is a **symmetric solution of dispersionless 2D Toda hierarchy** such that $\partial_0 F_{con}(t_0, \mathbf{t} = 0, \bar{\mathbf{t}} = 0) = -t_0 + t_0 \ln t_0 + const$

Double Hurwitz numbers

Let us consider a rational function in general position $\varphi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ of degree d with zeros of degrees $\Delta = [\mu_1, \dots, \mu_\ell]$ and poles of degrees $\bar{\Delta} = [\bar{\mu}_1, \dots, \bar{\mu}_{\bar{\ell}}]$. The general position means, that all other critical points are simple and have different images $z_1, \dots, z_k$. The number $H_0(\Delta, \bar{\Delta})$ of such functions is finite and don't depend from $z_1, \dots, z_k$. It is called *double Hurwitz number* of genus 0. They describe topological properties of moduli spaces of complex algebraic curves and they importent for modern mathematical physics.

It is follow from A.Okounkov theorem (2000), that generating function for double Hurwitz number

$$F(t_0, \mathbf{t}, \bar{\mathbf{t}}) = \frac{t_0^3}{6} + \sum_{|\Delta|=|\bar{\Delta}|} \frac{e^{|\Delta| t_0} H_0(\Delta, \bar{\Delta})}{(\ell(\Delta)+\ell(\bar{\Delta})-2)!} \prod_{i=1}^{\ell(\Delta)} \mu_i t_{\mu_i} \prod_{i=1}^{\ell(\bar{\Delta})} \bar{\mu}_i \bar{t}_{\bar{\mu}_i},$$

is a **symmetric solution of dispersionless 2D Toda hierarchy.**

Dispersionless 2D Toda hierarchy

was defined by K.Takasaki and T.Takebe in 1994 year for application in mathematical physics. Late was found its application in topology, complex analysis, and algebraic geometry.

Dispersionless 2D Toda hierarchy is equations on a function $F = F(t_0, \mathbf{t}, \bar{\mathbf{t}})$ from a variable t_0 and two infinite sets of variables , $\mathbf{t} = \{t_1, t_2, \dots\}$, $\bar{\mathbf{t}} = \{\bar{t}_1, \bar{t}_2, \dots\}$. Let us put $\partial_0 = \partial/\partial t_0$, $\partial_k = \partial/\partial t_k$, $\bar{\partial}_k = \partial/\partial \bar{t}_k$ and

$$D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_k , \qquad \bar{D}(\bar{z}) = \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \bar{\partial}_k ,$$

The hierarchy is presented by equations on formal Laurent series

$$(z - \xi) \exp\big(D(z)D(\xi)F\big) = z \exp\big(-\partial_0 D(z)F\big) - \xi \exp\big(-\partial_0 D(\xi)F\big), \quad (1)$$

$$(\bar{z} - \bar{\xi}) \exp\big(\bar{D}(\bar{z})\bar{D}(\bar{\xi})F\big) = \bar{z} \exp\big(-\partial_0 \bar{D}(\bar{z})F\big) - \bar{\xi} \exp\big(-\partial_0 \bar{D}(\bar{\xi})F\big), \quad (2)$$

$$1 - \exp\big(-D(z)\bar{D}(\bar{\xi})F\big) = \frac{1}{z\bar{\xi}} \exp\big(\partial_0(\partial_0 + D(z) + \bar{D}(\bar{\xi}))F\big). \quad (3)$$

where z and ξ are auxiliary complex variables.

Formal solutions

We consider *formal solutions* i.e. solutions in form of formal Taylor series

$$F(t_0, \mathbf{t}, \bar{\mathbf{t}}) =$$

$$\sum_{\substack{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_\ell > 0 \\ \bar{\mu}_1 \geq \bar{\mu}_2 \geq \ldots \geq \bar{\mu}_{\bar{\ell}} > 0}} F(\mu_1, \mu_2, \ldots, \mu_\ell | \bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_{\bar{\ell}} | t_0) t_{\mu_1} t_{\mu_2} \ldots t_{\mu_k} \bar{t}_{\bar{\mu}_1} \bar{t}_{\bar{\mu}_2} \ldots \bar{t}_{\mu_{\bar{\ell}}}.$$

The indexes $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_\ell > 0$ and $\bar{\mu}_1 \geq \bar{\mu}_2 \geq \ldots \geq \bar{\mu}_{\bar{\ell}} > 0$ form Young diagrammes Δ and $\bar{\Delta}$.
Denote by $t_\Delta = t_{\mu_1} t_{\mu_2} \ldots t_{\mu_k}$, and $t_{\bar{\Delta}} = \bar{t}_{\bar{\mu}_1} \bar{t}_{\bar{\mu}_2} \ldots \bar{t}_{\mu_{\bar{\ell}}}$. Then

$$F(t_0, \mathbf{t}, \bar{\mathbf{t}}) = F(\emptyset|\emptyset|t_0) + \sum_{\Delta} F(\Delta|\emptyset|t_0) t_\Delta + \sum_{\bar{\Delta}} F(\emptyset|\bar{\Delta}|t_0) \bar{t}_{\bar{\Delta}}$$

$$+ \sum_{\Delta, \bar{\Delta}} F(\Delta|\bar{\Delta}|t_0)\, t_\Delta \bar{t}_{\bar{\Delta}}.$$

Symmetric solutions

Denote by $g|_{t_0}(t_0) = g(t_0, 0, 0)$ the restriction of $g(t_0, \mathbf{t}, \bar{\mathbf{t}})$ on t_0. In applications it needed usual *symmetric solutions* $F(t_0, \mathbf{t}, \bar{\mathbf{t}})$ of dispersionless 2D Toda hierarchy. This means $\partial_k F|_{t_0} = \bar{\partial}_k F|_{t_0} = 0$ for $k > 0$.

Theorem 1 (S.Nat., A.Zabrodin 2013). Any formal symmetric solution of 2D dispersionless Toda hierarchy F is defined by any function $\Phi(t_0)$ and it is equal

$$F = \Phi(t_0) + \sum_{i>0}^{\infty} if^i t_i \bar{t}_i +$$

$$\sum_{|\Delta|=|\bar{\Delta}|} \sum_{\substack{s_1+\ldots+s_m=|\Delta| \\ r_1+\ldots+r_m=\ell(\Delta)+\ell(\bar{\Delta})-2>0}} N_{(\Delta|\bar{\Delta})} \binom{s_1 \ldots s_m}{r_1 \ldots r_m} \partial_0^{r_1}(f^{s_1}) \ldots \partial_0^{r_m}(f^{s_m})\, t_\Delta \bar{t}_{\bar{\Delta}}, \quad (4)$$

where $f(t_0) = \exp(\Phi'')(t_0)$. The sum is given by positive integer indexes, $|\Delta|$ is the order of Young diagram Δ and $\ell(\Delta)$ is the number of rows Δ.

Calculation of $N_{(\Delta|\bar{\Delta})}\begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix}$

We find also recursion algorithm for calculation of $N_{(\Delta|\bar{\Delta})}\begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix}$. It consists of several steps.

1. **Denote** by $P_{ij}(r_1, \dots, r_m)$ is the number of sequences of positive integers $(i_1, \dots, i_m)$, $(j_1, \dots, j_m)$ such that $i_1 + \dots + i_m = i$, $j_1 + \dots + j_m = j$ and $r_k = i_k + j_k$.

Put $T_{ij}(p_1, \dots, p_m) =$

$$\sum_{\substack{k>0,\, n_i>0 \\ n_1+\cdots+n_k=m}} \frac{(-1)^{m+1}}{k\, n_1! \dots n_k!} P_{ij}\left(\sum_{i=1}^{n_1} p_i, \sum_{i=n_1+1}^{n_1+n_2} p_i, \ \dots, \sum_{i=n_1+\dots+n_{k-1}+1}^{m} p_i\right).$$

2. **Define**

$$T_{i_1 i_2}\begin{pmatrix} s_1 \dots s_m \\ \ell_1 \dots \ell_m \end{pmatrix} = \begin{cases} T_{i_1 i_2}(s_1, \dots, s_m), & \text{if } \ell_1 = \dots = \ell_m = 1 \\ \\ 0 & \text{otherwise,} \end{cases}$$

and

$$T_{i_1 \dots i_k}\begin{pmatrix} s_1 \dots s_m \\ \ell_1 \dots \ell_m \end{pmatrix} = \sum_{1 \leqslant i \leqslant j \leqslant m} \frac{\ell!}{(\ell_i - 1)! \dots (\ell_j - 1)!} \times$$

$$T_{i_1 \dots i_{k-1}}\begin{pmatrix} s_1 \dots s_{i-1}\ s\ s_{j+1} \dots s_m \\ \ell_1 \dots \ell_{i-1}\ \ell\ \ell_{j+1} \dots \ell_m \end{pmatrix} T_{s, i_k}(s_i, s_{i+1}, \dots, s_j),$$

where

$$s = s_i + s_{i+1} + \dots + s_j - i_k > 0, \quad \ell = (\ell_i - 1) + \dots + (\ell_j - 1) > 0.$$

3. **Consider** $\tilde{N}_{\binom{i_1 \dots i_k}{\bar{i}_1 \dots \bar{i}_{\bar{k}}}} \begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix} =$

$$\frac{i_1 \dots i_k \, \bar{i}_1 \dots \bar{i}_{\bar{k}}}{s_1 \dots s_m} \sum T_{i_1 \dots i_k} \begin{pmatrix} s_1 \dots s_m \\ r_1 - n_1 + 1 \dots r_m - n_m + 1 \end{pmatrix},$$

where the summation is carried over all representation of the set $\{\bar{i}_1, ..., \bar{i}_{\bar{k}}\}$ as a union of non-intersecting non-empty subsequences $\{b^j_1, ..., b^j_{n_j}\} \subset \{\bar{i}_1, ..., \bar{i}_{\bar{k}}\}$ such that $b^j_1 + \dots + b^j_{n_j} = s_j$ and $j = 1, \dots, m$.

Put

$$N_{(\Delta|\bar{\Delta})} \begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix} = \frac{1}{\sigma(\Delta)\sigma(\bar{\Delta})} \tilde{N}_{\binom{\mu_1 \dots \mu_k}{\bar{\mu}_1 \dots \bar{\mu}_{\bar{k}}}} \begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix}, \quad (5)$$

where Δ, $\bar{\Delta}$ are Young diagrams with rows $[\bar{\mu}_1, \dots, \bar{\mu}_\ell]$ and $[\mu_1, \dots, \mu_\ell]$ and $\sigma(\Delta)$, $\sigma(\bar{\Delta})$ are orders of the automorphism groups of rows for the Young diagrams Δ, $\bar{\Delta}$.

Examples

Example 1. If $\Delta = [\mu_1, \dots, \mu_l]$, $\bar\Delta = [\bar\mu_1, \dots, \bar\mu_{\bar l}]$ and $d = |\Delta| = |\bar\Delta|$, then

$$N_{(\Delta|\bar\Delta)}\binom{d}{\ell(\Delta)+\ell(\bar\Delta)-2} = \frac{\rho(\Delta)\rho(\bar\Delta)}{d\sigma(\Delta)\sigma(\bar\Delta)},$$

where $\rho(\Delta) = \mu_1 \cdots \mu_l$. For other cases $N_{(\Delta|\bar\Delta)}\binom{s}{r} = 0$.

Example 2.

$$N_{\binom{i_1 i_2}{\bar i_1 \bar i_2}}\binom{\bar i_1 \bar i_2}{1 1} = N_{\binom{i_1 i_2}{\bar i_1 \bar i_2}}\binom{\bar i_2 \bar i_1}{1 1} = -\frac{i_1 i_2}{2\sigma([i_1, i_2])\sigma([\bar i_1, \bar i_2])}\min\{i_1, i_2, \bar i_1, \bar i_2\})$$

For other cases $N_{\binom{i_1 i_2}{\bar i_1 \bar i_2}}\binom{s_1 s_2}{r_1 r_2} = 0$.

Application to Hurwitz numbers

Okounkov's theorem says that

$$F(t_0, \mathbf{t}, \bar{\mathbf{t}}) = \frac{t_0^3}{6} + \sum_{|\Delta|=|\bar{\Delta}|} \frac{e^{|\Delta| t_0} H_0(\Delta, \bar{\Delta})}{(\ell(\Delta)+\ell(\bar{\Delta})-2)!} \prod_{i=1}^{\ell(\Delta)} \mu_i t_{\mu_i} \prod_{i=1}^{\ell(\bar{\Delta})} \bar{\mu}_i \bar{t}_{\bar{\mu}_i},$$

is a symmetric formal solutions of dispersionless 2D Toda hierarchy with $\Phi(t_0) = \frac{t_0}{6}$ and $f(t_0) = \exp(t_0)$. Thus Theorem 1 gives

Theorem 2 (S.Nat., A.Zabrodin 2013). Double Hurwitz numbers of genus 0 are

$$H_0(\Delta|\bar{\Delta}) = \frac{(\ell(\Delta) + \ell(\bar{\Delta})-2)!}{\rho(\Delta)\,\rho(\bar{\Delta})} \sum s_1^{r_1} \dots s_m^{r_m} N_{(\Delta|\bar{\Delta})} \begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix},$$

where the sum is carried over all matrixes $\begin{pmatrix} s_1 \dots s_m \\ r_1 \dots r_m \end{pmatrix}$ such that $s_1 + \dots + s_m = |\Delta|$ and $r_1 + \dots + r_m = \ell(\Delta)+\ell(\bar{\Delta})-2$.

In particulary examples 1 and 2 give

Example 3.
The double Hurwitz numbers of polynomials are

$$H_0(\Delta|[n]) = \frac{(\ell(\Delta)-1)!}{\sigma(\Delta)}\, n^{\ell(\Delta)-2}.$$

Example 4.
The double Hurwitz numbers of simplest Laurent polynomials are

$$H_0([i_1,i_2]|[\bar{i}_1,\bar{i}_2]) = 2\frac{d-\min\{i_1,i_2,\bar{i}_1,\bar{i}_2\}}{(1+\delta_{i_1 i_2})(1+\delta_{\bar{i}_1 \bar{i}_2})}.$$

These formulas were at first found by integration on compactification of moduli spaces of algebraic curves (D.Zvonkine 1997; I.Goulden, D.Jackson, Vakil 2005; S.Shadrin, M.Shapiro, A.Vainshtein 2008).

Application to Riemann theorem

As far as F_{con} is a symmetric solutions to dispersionless 2D Toda hierarchy, with $\Phi(t_0) = \frac{1}{2}t_0^2 \ln t_0 - \frac{3}{4}t_0^2$ and $f(x) = t_0$, then

Theorem 3 (S.Nat. 2004). The conformal map from Q to Λ is

$$w_Q(z) = z \exp\left(\left(-\frac{1}{2}\partial_0^2 - \partial_0 D(z)\right) F_{con}\right)(t_0, \mathbf{t}(Q), \bar{\mathbf{t}}(Q)),$$

where

$$F_{con}(t_0, \mathbf{t}, \bar{\mathbf{t}}) = \frac{1}{2}t_0^2 \ln t_0 - \frac{3}{4}t_0^2 + \sum_{i>0}^{\infty} i t_0^i t_i \bar{t}_i +$$

$$\sum_{|\Delta|=|\bar{\Delta}|} \sum_{\substack{s_1+\ldots+s_m=|\Delta| \\ r_1+\ldots+r_m=\ell(\Delta)+\ell(\bar{\Delta})-2>0}} N_{(\Delta|\bar{\Delta})} \begin{pmatrix} s_1 \ldots s_m \\ r_1 \ldots r_m \end{pmatrix} \partial_0^{r_1}(t_0^{s_1}) \ldots \partial_0^{r_m}(t_0^{s_m})\, t_\Delta \bar{t}_{\bar{\Delta}}.$$